1. Let $\gamma \in \mathbb{C}$ and consider the function $f(z) = z^\gamma = e^{\gamma \log(z)}$. Show that it is holomorphic on the domain $\mathcal{U} = \mathbb{C} \setminus \{z : \operatorname{Im}(z) = 0, \operatorname{Re}(z) \leqslant 0\}$ and that its derivative satisfies

$$f'(z) = \gamma z^{\gamma-1}.$$

2. Consider the function $f(z) = \log(1+z^2)$. What is its domain of definition? What is the biggest open subset of $\mathbb{C}$ on which $f(z)$ is holomorphic? Compute its derivative.

3. *(Matrix representation of complex numbers)* For any complex number $z = x + yi$, $x, y \in \mathbb{R}$, define the 2×2 matrix $M(z)$ by

$$M(z) \doteq \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Show that

$$M(z_1+z_2) = M(z_1)+M(z_2), \quad M(z_1 \cdot z_2) = M(z_1) \cdot M(z_2) \text{ and, for } z \neq 0 : M(z^{-1}) = \big(M(z)\big)^{-1}.$$

4. Let $z = x + yi \to f(z) = u(x,y) + v(x,y)i$ be an entire function. Define the following vector fields on $\mathbb{R}^2$:

$$F(x,y) = \begin{pmatrix} u(x,y) \\ -v(x,y) \end{pmatrix}, \quad G(x,y) = \begin{pmatrix} v(x,y) \\ u(x,y) \end{pmatrix}.$$

Compute the divergence and the curl of F and G.

5. Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Show that if $\operatorname{Re}(f)$ is constant, then f is also constant. *(Hint: Use the Cauchy–Riemann equations.)*. Similarly, show that if $|f|$ is constant, then f is constant.

6. Compute the following contour integrals:

(a) $\int_\gamma (z^2 + 1)\, dz$, where $\gamma = [1, 1+i]$ (line segment connecting 1 with $1+i$).

(b) $\int_\gamma \operatorname{Re}(z^2)\, dz$, where $\gamma = \{z : |z| = 1\}$ oriented counter-clockwise.

(c) $\int_\gamma \frac{z+1}{z^2+2}\, dz$, where $\gamma = \{z : |z| = 1\}$ oriented clockwise.

Solutions

1. **(a)** Holomorphicity:

The logarithm function $f(z) = \log(z)$ is holomorphic on $\mathbb{C} \setminus \{z : \mathrm{Im}(z) = 0,\, \mathrm{Re}(z) \leqslant 0\}$ (the complex plane minus the non-positive real axis). The function $g(z) = e^{\gamma z}$ is entire. Therefore, $e^{\gamma \log(z)}$ is holomorphic on $\mathbb{C} \setminus \{z : \mathrm{Im}(z) = 0,\, \mathrm{Re}(z) \leqslant 0\}$ as the composition $g \circ f(z)$.

(b) Derivative:

Using the chain rule $\big(g(f(z))\big)' = g'(f(z)) \cdot f'(z)$, we compute:

$$f(z) = e^{\gamma \log(z)} \implies f'(z) = e^{\gamma \log(z)} \cdot \big(\gamma \log(z)\big)' = e^{\gamma \log(z)} \cdot \frac{\gamma}{z} = \gamma z^{\gamma - 1}$$

2. Recall that the domain of *definition* of $\log(z)$ is $\mathbb{C} \setminus \{0\}$, while the domain on which it is holomorphic is $\mathbb{C} \setminus \{z : \mathrm{Im}(z) = 0,\, \mathrm{Re}(z) \leqslant 0\}$. So, for $f(z) = \log(1 + z^2)$:

- Domain of definition: We must have $1 + z^2 \neq 0$, so the domain is $\mathbb{C} \setminus \{i, -i\}$.
- Domain of holomorphicity: We must have $1 + z^2 \in \mathbb{C} \setminus \{z : \mathrm{Im}(z) = 0,\, \mathrm{Re}(z) \leqslant 0\}$. If $z = x + yi$, then

$$1 + z^2 = 1 + x^2 - y^2 + 2xyi,$$

so $1 + z^2 \in \{z : \mathrm{Im}(z) = 0,\, \mathrm{Re}(z) \leqslant 0\}$ if and only if

$$2xy = 0 \text{ and } 1 + x^2 - y^2 \leqslant 0.$$

The first relation above yields $x = 0$ or $y = 0$. If $y = 0$, the second relation becomes $1 + x^2 \leqslant 0$, which is impossible (recall that x, y are real). For $x = 0$, the second relation becomes

$$y^2 \geqslant 1 \Leftrightarrow y \in (-\infty, -1] \cup [1, +\infty).$$

So $f(z)$ is holomorphic for $z \in \mathbb{C} \setminus \{yi :\, y \in (-\infty, -1] \cup [1, +\infty)\}$, i.e. on $\mathbb{C}$ without two imaginary half-lines, one emanating from $+i$ towards $+\infty i$ and the other from $-i$ towards $-\infty i$.

3. **(a)** Addition Property:

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

$$M(z_1 + z_2) = \begin{pmatrix} x_1 + x_2 & -(y_1 + y_2) \\ y_1 + y_2 & x_1 + x_2 \end{pmatrix}$$

$$M(z_1) + M(z_2) = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} + \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 & -(y_1 + y_2) \\ y_1 + y_2 & x_1 + x_2 \end{pmatrix}$$

Thus,

$$M(z_1 + z_2) = M(z_1) + M(z_2)$$

(b) Multiplication Property:

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + (x_1y_2 + x_2y_1)i$$

$$M(z_1 \cdot z_2) = \begin{pmatrix} x_1x_2 - y_1y_2 & -(x_1y_2 + x_2y_1) \\ x_1y_2 + x_2y_1 & x_1x_2 - y_1y_2 \end{pmatrix}$$

$$M(z_1) \cdot M(z_2) = \begin{pmatrix} x_1 & -y_1 \\ y_1 & x_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 & -(x_1y_2 + x_2y_1) \\ x_1y_2 + x_2y_1 & x_1x_2 - y_1y_2 \end{pmatrix}$$

Thus,

$$M(z_1 \cdot z_2) = M(z_1) \cdot M(z_2)$$

(c) Inverse Property:

$$z^{-1} = \frac{x - yi}{x^2 + y^2}$$

$$M(z^{-1}) = \begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

$$M(z)^{-1} = \frac{1}{x^2 + y^2} \begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$$

Thus,

$$M(z^{-1}) = M(z)^{-1}$$

4. Recall that, for a vector field $V(x, y) = (V_1(x, y), V_2(x, y))$, we define

$$\text{div}(V) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} \quad \text{and} \quad \text{curl}(V) = \frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y}.$$

So, using the Cauchy–Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

we immediately calculate:

$$\text{div}(F) = \frac{\partial u}{\partial x} + \frac{\partial(-v)}{\partial y} = 0,$$

$$\mathrm{curl}(F) = \frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y} = 0,$$

$$\mathrm{div}(G) = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0,$$

$$\mathrm{curl}(G) = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0.$$

5. **(a)** If $\mathrm{Re}(f)$ is constant: Let $f(z) = u(x,y) + iv(x,y)$, where u is constant.
By the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Since u is constant, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$.
Therefore, $\frac{\partial v}{\partial y} = 0$ and $\frac{\partial v}{\partial x} = 0$, implying that v is also constant.
Hence, $f(z)$ is constant.

(b) If $|f|$ is constant: Let $|f(z)| = c$, so $u^2 + v^2 = c^2$. If $c = 0$, then $u = v = 0$, which implies that f is constant in this case. So it remains to treat the case when $c \neq 0$; in this case, at every point (x,y), at least one of $u(x,y)$ and $v(x,y)$ is non-zero. Differentiating both sides with respect to x and y:

$$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$$

or, after dividing by 2:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0 \quad \text{and} \quad u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$$

The Cauchy–Riemann equations read $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$, so we get from the above:

$$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0 \quad \text{and} \quad u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0.$$

As we said in the beginning, at every point, at least one of the u, v is non-zero. Assume, without loss of generality that $u \neq 0$ (the argument will be analogous if $v \neq 0$): Dividing then the above with u, we get

$$\frac{\partial u}{\partial x} = \frac{v}{u}\frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{v}{u}\frac{\partial u}{\partial x}.$$

Using the first equation for the right hand side of the second, we get:

$$\frac{\partial u}{\partial y} = -\frac{v}{u}\frac{\partial u}{\partial x} = -\Big(\frac{v}{u}\Big)^2\frac{\partial u}{\partial y} \Leftrightarrow \Big(1 + \Big(\frac{v}{u}\Big)^2\Big)^2\frac{\partial u}{\partial y} = 0 \Leftrightarrow \frac{\partial u}{\partial y} = 0.$$

This implies from the first equation that also $\frac{\partial u}{\partial x} = 0$, so u is constant. Similarly, v is constant. Therefore, $f(z)$ is constant.

6. (a)

$$\int_\gamma (z^2 + 1)\, dz$$

Parametrize γ: $\gamma(t) = 1 + it$, $t \in [0, 1]$. So $\gamma'(t) = i$.
The integral becomes:

$$\int_0^1 [(1 + it)^2 + 1] \cdot i\, dt = i\int_0^1 (1 + 2it - t^2 + 1)\, dt = i\int_0^1 (2 + 2it - t^2)\, dt$$
$$= i\left[2t + it^2 - \frac{t^3}{3}\right]_0^1 = i\left(2 + i - \frac{1}{3}\right) = i\left(\frac{5}{3} + i\right) = -1 + \frac{5i}{3}$$

(b)

$$\int_\gamma \mathrm{Re}(z^2)\, dz$$

Parametrize γ: $\gamma(\theta) = e^{i\theta}$, $\theta \in [0, 2\pi]$. $\gamma'(\theta) = ie^{i\theta}$.
$\mathrm{Re}(z^2) = \mathrm{Re}(e^{2i\theta}) = \cos(2\theta)$.
The integral becomes:

$$\int_0^{2\pi} \cos(2\theta) \cdot ie^{i\theta}\, d\theta = i\int_0^{2\pi} \cos(2\theta)e^{i\theta}\, d\theta$$

Using Euler's formula:

$$\cos(2\theta) = \frac{e^{2i\theta} + e^{-2i\theta}}{2}$$

The integral becomes:

$$i\int_0^{2\pi} \frac{e^{3i\theta} + e^{-i\theta}}{2}\, d\theta = \frac{i}{2}\left(\int_0^{2\pi} e^{3i\theta}\, d\theta + \int_0^{2\pi} e^{-i\theta}\, d\theta\right) = \frac{i}{2}\left(\left[\frac{e^{3i\theta}}{3i}\right]_{\theta=0}^{2\pi} + \left[\frac{e^{-i\theta}}{-i}\right]_{\theta=0}^{2\pi}\right) = 0.$$

(c)

$$\int_\gamma \frac{z+1}{z^2+2}\,dz$$

Parametrize γ: $\gamma(\theta) = e^{-i\theta}$, $\theta \in [0, 2\pi]$ (clockwise direction).
The roots of $z^2 + 2 = 0$ are $z = \pm i\sqrt{2}$, both outside the unit circle.
By Cauchy's theorem, since $\frac{z+1}{z^2+2}$ is holomorphic in an open disk containing the unit circle, the integral result is 0.